MPIH–V18–1996
hep-ph/9605409

Convergence of Discretized Light Cone Quantization in the small mass limit

Brett van de Sande

Max Planck Institut für Kernphysik,
Postfach 10.39.80, D-69029 Heidelberg, Germany

Abstract

I discuss the slow convergence of Discretized Light Cone Quantization (DLCQ) in the small mass limit and suggest a solution.

1 Introduction

"Light-front field theory" is an attempt to apply Hamiltonian techniques to relativistic field theory; the idea is to quantize a theory on a surface of constant $x^+ = (x^0 + x^3)/\sqrt{2}$ and calculate eigenstates of the invariant mass squared operator $P^\mu P_\mu$. In order to partially regulate a theory, one can simply impose (anti-)periodic boundary conditions in the x^- coordinate; this is known as "Discretized Light Cone Quantization" (DLCQ) [1] since the momenta conjugate to x^- become discrete. One can use DLCQ to construct a basis of states for use in numerical calculations.

Why would one want to use DLCQ in a large scale numerical calculation? Ideally, one would employ a basis of polynomial-type wavefunctions to solve the bound state equation [2]. With such a basis, convergence of the eigenvalues is typically exponentially fast. However, the practitioner faces several difficulties in applying this approach to a many particle calculation. In general, the resulting basis is not orthogonal and the resulting Hamiltonian matrix is no longer very sparse. Second, matrix elements become difficult to compute since they involve complicated integrals. It is not clear that the faster convergence of such a basis can compensate for these added difficulties.

In contrast, a DLCQ basis is orthogonal and the corresponding Hamiltonian matrix is sparse. Also, matrix elements are easily calculated; no integrals need to be evaluated. This is important since the time required to calculate matrix elements is currently the limiting factor in large calculations. Finally, as a regulator, DLCQ has an appealing physical interpretation in terms of (anti-)periodic boundary conditions.

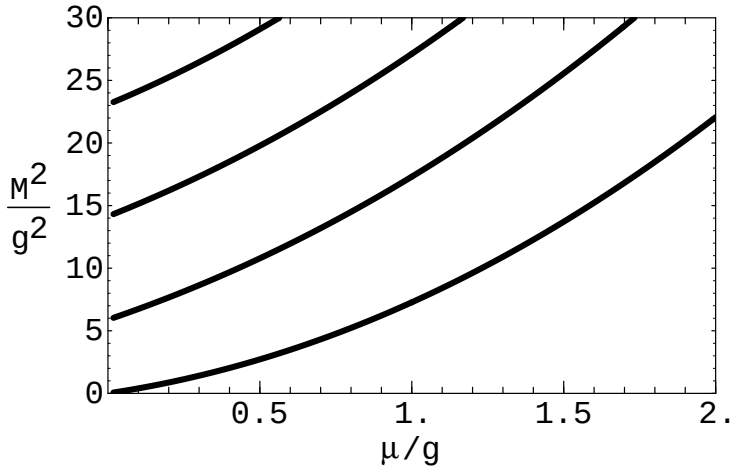

Figure 1: Lowest eigenvalues of the 't Hooft equation as a function of mass μ calculated using a polynomial wavefunction basis [2].

However, DLCQ has one distinct disadvantage: it converges quite slowly in the small mass limit. This is especially of concern for the transverse lattice [3, 4] where the continuum limit is precisely the small mass limit. In particular, the lowest eigenvalue and measurements of the string tension are strongly affected by this slow convergence. As we shall see, the source of this slow convergence comes from the end-point behavior of the wavefunction which is handled poorly by DLCQ. Using the 't Hooft equation as an example, we will study this behavior and suggest a solution that can be easily applied to other theories and large numerical calculations.

2 The 't Hooft equation

The 't Hooft equation [5] is the bound state equation for two quarks in the large N limit of (1+1)-dimensional QCD:

$$M^2\Psi(x) = \mu^2\Psi(x)\left(\frac{1}{x}+\frac{1}{1-x}\right) + g^2\int_0^1 dy\,\frac{\Psi(x)-\Psi(y)}{(x-y)^2} \tag{1}$$

where M is the Lorentz invariant mass eigenvalue associated with eigenfunction $\Psi(x)$. A Cauchy principle value prescription is assumed throughout. The spectrum is shown in Fig. 1. Following 't Hooft, let us analyze the endpoint behavior of the wavefunction. Assume

$$\lim_{x\to 0}\Psi(x) \propto x^\beta\ . \tag{2}$$

For small x, the bound state equation (1) becomes

$$0 = x^{\beta-1}\mu^2 - g^2x^{\beta-1} - g^2\int_0^1 dy\,\frac{y^\beta}{(x-y)^2}\,(1+O(y)) + O\big(x^\beta\big)\ . \tag{3}$$

Substituting $z = y/x$, we obtain, in the $x \to 0$ limit,

$$0 = \frac{\mu^2}{g^2} - 1 - \int_0^{1/x} dz\, \frac{z^\beta}{(1-z)^2} \tag{4}$$

$$= \frac{\mu^2}{g^2} - 1 + \pi\beta \cot(\pi\beta)\ . \tag{5}$$

Convergence of the integral (4) implies $0 < \beta < 1$. For small mass μ

$$\beta = \frac{\mu\sqrt{3}}{g\pi}\left(1 - \frac{\mu^2}{10g^2} + \cdots\right)\ . \tag{6}$$

In addition, one can derive an expression for the lowest eigenvalue in this limit

$$M^2 = \frac{2\pi g\mu}{\sqrt{3}} + O\big(\mu^2\big)\ . \tag{7}$$

3 Analysis and discussion

Exact solutions of the 't Hooft equation (1) are not known and one must resort to numerical techniques. For the purpose of comparison, we will introduce a basis of wavefunctions of the form $x^\beta(1-x)^\beta P_n(x)$ where $P_n(x)$ is some polynomial of order n and β is given by Eqn. (5). Such a basis provides very quick numerical convergence [2]. Later, when I quote "exact eigenvalues," I will use results from such a calculation.

Alternatively, we can use DLCQ to solve the bound state equation. We discretize the momentum fraction x as i/K where $i \in \{1/2, 3/2, \ldots, K-1/2\}$ for integer-valued "harmonic resolution" K. Physically, this discretization corresponds to applying anti-periodic boundary conditions in the x^- direction where $K \to \infty$ is the continuum limit. The DLCQ version of the bound state equation (1) is

$$M^2\Psi_i = \mu^2 K\Psi_i\left(\frac{1}{i} + \frac{1}{K-i}\right) + g^2 K \sum_{j\neq i} \frac{\Psi_i - \Psi_j}{(i-j)^2}\ . \tag{8}$$

Although a proof is lacking, numerical evidence suggests that the DLCQ eigenvalues converge as $1/K^{2\beta}$. For example, in Fig. 2, we plot the lowest DLCQ eigenvalue as a function of K together with a numerical fit to

$$M^2 = 0.779141 - \frac{1.25943}{K^{2\beta}} + \frac{1.15609}{K^{4\beta}} - \frac{1.01505}{K^{6\beta}} + \cdots \tag{9}$$

where the "exact eigenvalue" (from a polynomial basis calculation) is $M^2 = 0.77914$.

Note that the DLCQ bound state equation does not make use of our knowledge of the endpoint behavior of the wavefunction. This is, in fact, the source of the slow DLCQ convergence. We can include the endpoint behavior by adding and subtracting a term from the 't Hooft equation:

$$M^2\Psi(x) = \mu^2\Psi(x)\left(\frac{1}{x} + \frac{1}{1-x}\right) + g^2\Psi(x)\,\mathrm{I}(x) + g^2\int_0^1 dy\, \frac{\Psi(x)\left(\frac{y(1-y)}{x(1-x)}\right)^\beta - \Psi(y)}{(x-y)^2} \tag{10}$$

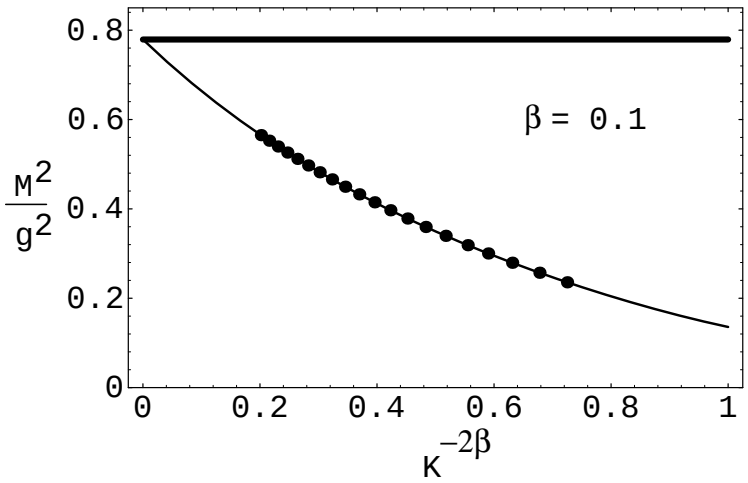

Figure 2: Convergence of the lowest eigenvalue with K for $\beta = 0.1$ ($\mu = 0.181981$). The dots are from DLCQ and the heavy line is the exact result.

where

$$\mathrm{I}(x) = \int_0^1 dy \, \frac{1 - \left(\frac{y(1-y)}{x(1-x)}\right)^\beta}{(x-y)^2} \, . \tag{11}$$

The DLCQ version of this equation is

$$M^2\Psi_i = \mu^2 K \Psi_i \left(\frac{1}{i} + \frac{1}{K-i}\right) + g^2 \Psi_i \, \mathrm{I}(i/K) + g^2 K \sum_{j \neq i} \frac{\Psi_i \left(\frac{j(K-j)}{i(K-i)}\right)^\beta - \Psi_j}{(i-j)^2} \tag{12}$$

where we evaluate the integral $\mathrm{I}(x)$ exactly.[1] Based on the endpoint analysis, the integrand in Eqn. (10) should vanish when both x and y are near one of the endpoints. Thus, the corresponding discretization (12) should have small errors in this region. We will refer to this form of the bound state equation as "improved DLCQ."

Let us see how this affects convergence of the spectrum. In Fig. 3, we see that the DLCQ results converge slowly and, in fact, have an incorrect functional form for M^2 versus μ in the small μ limit. On the other hand, the improved DLCQ eigenvalues are nearly identical to the exact result. However, this absence of finite K corrections is the result of a fortuitous accident: for small μ, the lowest eigenfunction is $\Psi(x) \approx x^\beta (1-x)^\beta$. Thus, the last term of Eqn. (12) is almost zero for all i and j and *all* finite K corrections vanish.

Examining an excited state, where the wavefunction has a more complicated form, will give us a better picture of what happens in the generic case. Ordinarily, DLCQ eigenvalues converge as $1/K^{2\beta}$ due to the endpoint regions $x \approx 0$ and $x \approx 1$ plus $1/K$ errors due to the $1/(x-y)^2$ singularity [6]. Thus, in Fig. 4, the DLCQ eigenvalues of the second excited state are fitted to powers of $1/K^{2\beta}$ plus powers

[1] In practice, I solve it numerically.

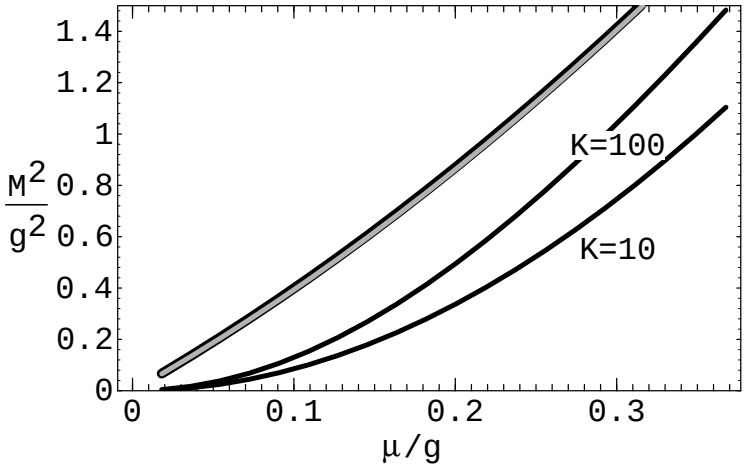

Figure 3: The lowest eigenvalue as a function of μ. The black lines are from DLCQ ($K = 10$ and 100) and the heavy line is the exact result. The improved DLCQ eigenvalue (gray line, $K = 10$) is almost identical to the the exact result.

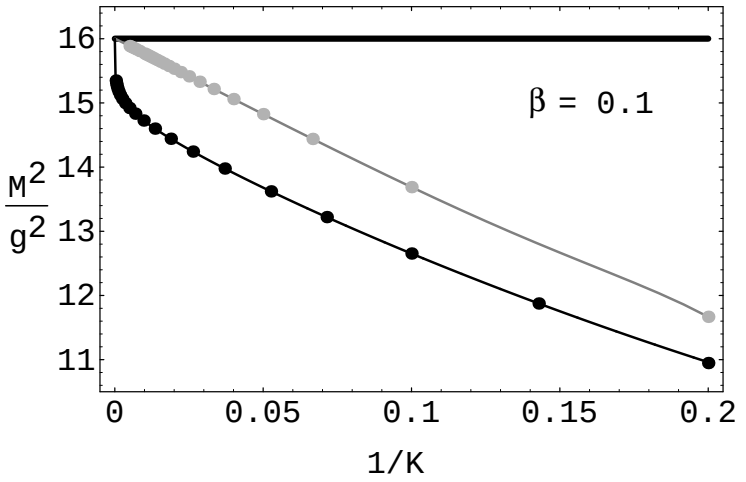

Figure 4: Convergence of the third lowest eigenvalue with K for $\beta = 0.1$ ($\mu = 0.181981$). The black dots are for DLCQ; the gray dots are for improved DLCQ; and the heavy line is the exact result.

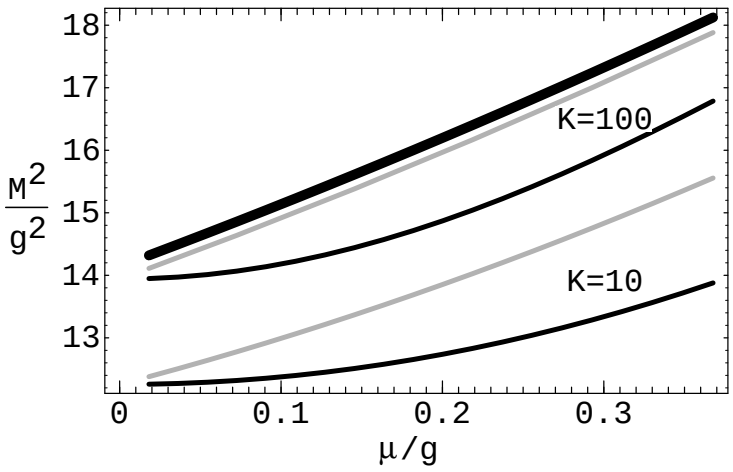

Figure 5: The third lowest eigenvalue as a function of μ. The black lines are from DLCQ ($K = 10$ and 100); the gray lines are improved DLCQ ($K = 10$ and 100); and the heavy line is the exact result.

of $1/K$,

$$M^2 = 16.0016 - \frac{3.86429}{K^{2\beta}} + \frac{3.88433}{K^{4\beta}} + \cdots - \frac{20.7358}{K} + \frac{19.0972}{K^2} \ , \tag{13}$$

while the improved DLCQ eigenvalues are fitted to powers of $1/K$,

$$M^2 = 16.0193 - \frac{24.7124}{K} + \frac{14.7355}{K^2} \ . \tag{14}$$

The exact result is $M^2 = 16.0016$. It is important to note that improved DLCQ removes the errors associated with the endpoint regions but does not significantly affect the $1/K$ errors. Finally, in Fig. 5 we see how convergence behaves as a function of μ for the second excited state.

So far, we have discussed the two particle bound state equation. The generalization to a DLCQ calculation with more particles is straightforward. For example, consider the following terms from the Schwinger model Hamiltonian [9, 2]:

$$\mu^2 \sum_{k=1/2}^{\infty} \frac{b_k^\dagger b_k + d_k^\dagger d_k}{k} + g^2 \sum_{k=1/2}^{\infty} \left(b_k^\dagger b_k + d_k^\dagger d_k \right) \sum_{n=1}^{k-1/2} \frac{1}{n^2} - g^2 \sum_{\substack{k,l,m,n=1/2 \\ k \neq l}}^{\infty} \frac{\delta_{k+m,l+n}}{(k-l)^2} \, b_k^\dagger b_l d_m^\dagger d_n \ . \tag{15}$$

In the two particle sector, this Hamiltonian produces the DLCQ version of the 't Hooft equation (8). The improved DLCQ version of (15) is

$$\mu^2 \sum_{k=1/2}^{\infty} \frac{b_k^\dagger b_k + d_k^\dagger d_k}{k} + g^2 \sum_{k,l=1/2}^{\infty} J_{k,l} \, b_k^\dagger b_k d_l^\dagger d_l - g^2 \sum_{\substack{k,l,m,n=1/2 \\ k \neq l}}^{\infty} \frac{\delta_{k+m,l+n}}{(k-l)^2} \, b_k^\dagger b_l d_m^\dagger d_n \tag{16}$$

where

$$J_{k,l} = J_{l,k} = \mathrm{I}\left(\frac{k}{k+l}\right) + \sum_{\substack{j=1/2 \\ j\neq k}}^{k+l-1/2} \frac{\left(\frac{j(k+l-j)}{kl}\right)^{\beta}}{(k-j)^2} \,. \tag{17}$$

We have replaced the second term of (15) by a four-point interaction since the improved DLCQ subtraction is sensitive to the momenta of both particles. The improved form (16) will remove the slow convergence from a many particle calculation.

4 Conclusion

There have been some past attempts to quantify the errors associated with DLCQ. The claim in Ref. [6] that the leading error is $1/K$ is certainly incorrect. In Ref. [7], periodic boundary conditions were applied to a somewhat different theory. In that case, a leading error of $1/\ln(K)$ seemed to explain the data.

There are a large number of DLCQ studies which have suffered from this slow convergence problem. This includes DLCQ studies of (1+1)-dimensional QCD [6, 8], studies of the Schwinger model [9, 10] (as noted in Ref. [11]), dimensionally reduced theories [7, 12], and the transverse lattice [4]. However, one should note that there are many cases where the slow convergence problem is not important: It is only of concern when μ is small but nonzero (the DLCQ subtraction is correct when $\mu = 0$) and $1/K$ errors can dominate in a many particle calculation (where the size of K is somewhat limited).

Acknowledgements

I would like to thank H.–C. Pauli and R. Bayer for useful discussions.

References

[1] H.–C. Pauli and S. Brodsky, Phys. Rev. D **32**, 1993 and 2001 (1985).

[2] Y. Mo and R. Perry, J. Comput. Phys. **108** 159 (1993); K. Harada, T. Sugihara, M. Taniguchi, amd M. Yahiro, Phys. Rev. D **49**, 4226 (1994); K. Harada, A. Okazaki, and M. Taniguchi, Phys. Rev. D **52**, 2429 (1995).

[3] W. A. Bardeen and R. B. Pearson, Phys. Rev. D **14**, 547 (1976); W. A. Bardeen, R. B. Pearson, and E. Rabinovici, Phys. Rev. D **21**, 1037 (1980).

[4] S. Dalley and B. van de Sande, Report Nos. DAMTP-96-21 and `hep-ph/9602291` (submitted to Phys. Lett.); B. van de Sande and S. Dalley, Report No. `hep-ph/9604268` (To be published in the proceedings of Orbis Scientæ1996).

[5] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).

[6] K. J. Hornbostel, Ph.D. Thesis, SLAC Report No. 333 (1988), unpublished; S. J. Brodsky, K. J. Hornbostel and H.–C. Pauli, Phys. Rev. D **41**, 3814 (1990).

[7] B. van de Sande and M. Burkardt, Phys. Rev. D **53**, 4628 (1996).

[8] M. Burkardt, Nucl. Phys. **A504**, 762 (1989); M. Heyssler and A. C. Kalloniatis, Phys. Lett. **B354**, 453 (1995).

[9] T. Eller, H.–C. Pauli, and S. J. Brodsky, Phys. Rev. D **35**, 1493 (1987).

[10] S. Elser, H.–C. Pauli, and A. Kalloniatis, Report Nos. MPIH-V17-1995 and `hep-th/9505069`.

[11] J.P. Vary, T.J. Fields, and H. J. Pirner in "Theory of Hadrons and Light Front QCD," edited by Stanisław D. Głazek. World Scientific, 1995. Report No. `hep-ph/9411263`.

[12] S. Dalley and I. R. Klebanov, Phys. Rev. D **47**, 2517 (1993); G. Bhanot, K. Demeterfi and I. R. Klebanov, Phys. Rev. D **48**, 4980 (1993); K. Demeterfi and I. R. Klebanov, Nucl. Phys. **B418**, 15 (1994).